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The approximation property for spaces of holomorphic functions on infinite-dimensional spaces I

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Abstract

For an open subset U of a locally convex space E, let $(H(U), \tau_0)$ denote the vector space of all holomorphic functions on U, with the compact-open topology. If E is a separable Fréchet space with the bounded approximation property, or if E is a (DFC)-space with the approximation property, we show that $(H(U), \tau_0)$ has the approximation property for every open subset U of E. These theorems extend classical results of Aron and Schottenloher. As applications of these approximation theorems we characterize the spectra of certain topological algebras of holomorphic mappings with values in a Banach algebra.

Keywords: Holomorphic function; Fréchet space; (DFC)-space; Approximation property

0. Introduction

Let $H(U)$ denote the vector space of all complex-valued holomorphic functions on a nonvoid open subset U of a complex locally convex space E. Let τ_0 denote the compact-open topology, and let τ_ω denote the Nachbin compact-ported topology on $H(U)$. In 1976 Aron and Schottenloher [2] gave necessary and sufficient conditions

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for $H(U)$ to have the approximation property when endowed with the topologies τ_0 or τ_ω. It follows from their results that if E is a Fréchet space or a (DFC)-space, and if U is a finitely Runge open subset of E, then $(H(U), \tau_0)$ has the approximation property if and only if E has the approximation property.

Every balanced open set, and every polynomially convex open set are finitely Runge. But what can be said when U is an arbitrary open set? Very little could be said in 1976, for that problem is closely connected with the Levi problem, envelopes of holomorphy and holomorphic approximation, which were not sufficiently understood at that time. But now we can say much more. In this paper we show that if E is a separable Fréchet space with the bounded approximation property, then $(H(U), \tau_0)$ has the approximation property for every open subset U of E. We also show that if U is an open subset of a (DFC)-space E, then $(H(U), \tau_0)$ has the approximation property if and only if E has the approximation property. As applications of our results we characterize the spectra of certain topological algebras of holomorphic mappings with values in Banach algebras or, more generally, with values in complete locally m-convex algebras. For additional applications we refer the reader to a recent article of Dineen [8].

This paper is devoted to the study of $(H(U), \tau_0)$. A subsequent paper will be devoted to the study of $(H(U), \tau_\omega)$.

Even if we are mainly interested in the study of open subsets of locally convex spaces, we deal here more generally with Riemann domains over locally convex spaces. The reason is that our proofs rely heavily on the machinery developed by Mujica [19] and Lourenço [16] for pseudoconvex Riemann domains over Fréchet spaces and (DFC)-spaces, respectively, and the fact, established by Alexander [1] and Schottenloher [25], that the envelope of holomorphy of an open subset of a locally convex space is a pseudoconvex Riemann domain over the same space.

1. The ε-product and the approximation property

The letters E, F denote locally convex spaces, always assumed complex and Hausdorff. $L(E;F)$ denotes the vector space of all continuous linear mappings from E into F. $L_c(E;F)$ denotes the vector space $L(E;F)$, endowed with the topology of uniform convergence on all convex, balanced, compact subsets of E. When $F = \mathbf{C}$, we write E' instead of $L(E;\mathbf{C})$, and E'_c instead of $L_c(E;\mathbf{C})$.

A main tool in this paper is the ε-product of Laurent Schwartz. Following Schwartz [27, Exposé no 8] (see also Schwartz [28, no. 1], Bierstedt [3, Definition 3.1] or Köthe [14, p. 242]) we let

$$E\varepsilon F = L_\varepsilon(E'_c;F)$$

denote the vector space $L(E'_c;F)$, endowed with the topology of uniform convergence on the equicontinuous subsets of E'. The space $E\varepsilon F$ is called the ε-product of E and F. Bierstedt's definition of the ε-product differs slightly from Schwartz' or Köthe's definition, but both definitions coincide in the case of quasi-complete locally convex spaces.

When U and V vary among the closed, convex, balanced 0-neighborhoods in E and F, respectively, then the sets

$$W(U^\circ; V) = \{T \in L(E'_c; F) : T(U^\circ) \subset V\}$$

form a 0-neighborhood base in $E\varepsilon F$.

Theorem 1.1 (Schwartz [27]). *If E and F are locally convex spaces, then the mapping $T \to T'$ is a topological isomorphism between the spaces $E\varepsilon F$ and $F\varepsilon E$.*

One can check that the mapping $T \to T'$ maps the set $W(U^\circ; V)$ onto the set $W(V^\circ; U)$.

Following Schwartz [27, Exposé no. 14] (see also Grothendieck [10, Chapitre I, no. 5]) we say that E has the *approximation property* if the identity mapping on E lies in the closure of $E' \otimes E$ in $L_c(E; E)$. Schwartz' definition of the approximation property (called the weak approximation property by Köthe [14, p. 232]) differs slightly from Grothendieck's definition, but both definitions coincide in the case of quasi-complete locally convex spaces. The following theorem summarizes results of Grothendieck [10, Chapitre I, no. 5] and Schwartz [27, Exposé no. 14] (see also Bierstedt [3, Satz 3.9] or Köthe [14, p. 243]).

Theorem 1.2 (Grothendieck [10], Schwartz [27]). *For a locally convex space E the following properties are equivalent*:

(1) E has *the approximation property.*

(2) $L_c(E; E) = \overline{E' \otimes E}$.

(3) $L_c(E; F) = \overline{E' \otimes F}$ *for every locally convex space F (equivalently for every Banach space F).*

(4) $L_c(F; E) = \overline{F' \otimes E}$ *for every locally convex space F.*

(5) $F\varepsilon E = \overline{F \otimes E}$ *for every locally convex space F (equivalently for every Banach space F).*

(6) $E\varepsilon F = \overline{E \otimes F}$ *for every locally convex space F (equivalently for every Banach space F).*

The equivalence of (1)–(4) (for every locally convex space F) is easily seen. Implication (4) $\Rightarrow$ (5) is immediate. Implication (5) $\Rightarrow$ (4) follows from the fact that $L_c(F; E)$ is a topological subspace of $F'_c\varepsilon E$. And equivalence (5) $\Leftrightarrow$ (6) follows from Theorem 1.1. The fact that each of conditions (3), (5) or (6) for every locally convex space F is equivalent to the corresponding condition for every Banach space F, was pointed out by Bierstedt and Meise [4, p. 100]. In the case of condition (3) or (6), the equivalence follows from the arguments of the proofs of two results from the book of Köthe [14, p. 233, (2), (3)]. In the case of condition (5), the equivalence follows from Theorem 1.1 and the corresponding equivalence in condition (6). We do not know whether a similar remark applies to condition (4).

Corollary 1.3. *A locally convex space E has the approximation property if E'_c has the approximation property.*

Proof. If E'_c has the approximation property, then it follows from Theorem 1.2(6) that

$$\overline{E' \otimes F} = E'_c \varepsilon F$$

for every locally convex space F. Since $L_c(E; F)$ is a topological subspace of $E'_c \varepsilon F$, it follows that

$$\overline{E' \otimes F} = L_c(E; F)$$

for every locally convex space F. Thus E has the approximation property, by Theorem 1.2(3).

2. The ε-product and spaces of holomorphic mappings

If U is a nonvoid open subset of a locally convex space E, then a mapping $f : U \to F$ is said to be *holomorphic* if it is continuous, and the function $\lambda \to \psi \circ f(a + \lambda b)$ is holomorphic on some open neighborhood of zero in $\mathbf{C}$ for each $a \in U$, $b \in E$ and $\psi \in F'$. $H(U; F)$ denotes the vector space of all holomorphic mappings from U into F. When $F = \mathbf{C}$, we write $H(U)$ instead of $H(U; \mathbf{C})$.

Let (X, ξ) be a *Riemann domain* over E, that is X is a Hausdorff topological space and $\xi : X \to E$ is a local homeomorphism. Let $\mathscr{V}(E)$ denote the collection of all open, convex, balanced neighborhoods of zero in E. A *section* of X is a continuous mapping $\sigma : A \to X$, where $A \subset E$, such that $\xi \circ \sigma = identity$ on A. For $S \subset X$ and $V \in \mathscr{V}(E)$, we write $S + V \subset X$ if for each $x \in S$ there is a section $\sigma_x : \xi(x) + V \to X$ such that $\sigma_x \circ \xi(x) = x$. Then we define $x + t = \sigma_x(\xi(x) + t)$ for every $x \in S$ and $t \in V$.

A mapping $f : X \to F$ is said to be *holomorphic* if for each $x \in X$ there is a section $\sigma : \xi(x) + V \to X$, with $V \in \mathscr{V}(E)$ and $\sigma \circ \xi(x) = x$, such that the mapping $f \circ \sigma$ is holomorphic on $\xi(x) + V$. $H(X; F)$ denotes the vector space of all holomorphic mappings from X into F. When $F = \mathbf{C}$, we write $H(X)$ instead of $H(X; \mathbf{C})$. Let τ_0 denote the compact-open topology on $H(X; F)$. We refer to the books of Dineen [7] or Mujica [20] for background information on infinite dimensional complex analysis.

We recall that a Hausdorff topological space X is said to be a *k-space* if a set $U \subset X$ is open in X whenever $U \cap K$ is open in K for each compact set $K \subset X$. Every metric space is a k-space. An open subset of a k-space is also a k-space. Hence it follows that if X is a Riemann domain over a locally convex space E, and E is a k-space, then X is also a k-space.

The following results are due to Schottenloher [23].

Lemma 2.1 (Schottenloher [23]). *Let (X, ξ) be a Riemann domain over a locally convex space E, and let F be a quasi-complete locally convex space.*

(a) *For each* $f \in H(X;F)$ *let* S_f *denote the continuous linear mapping*

$$\psi \in F'_c \to \psi \circ f \in (H(X), \tau_0).$$

Then the mapping

$$f \in (H(X;F), \tau_0) \to S_f \in F\varepsilon(H(X), \tau_0)$$

is linear, continuous and injective.

(b) *Let* $\varepsilon_X : x \in X \to \varepsilon_x \in (H(X), \tau_0)'$ *denote the evaluation mapping, that is* $\varepsilon_x(f) = f(x)$ *for every* $f \in H(X)$. *If* E *is a k-space, then*

$$\varepsilon_X \in H(X; (H(X), \tau_0)'_c).$$

Theorem 2.2 (Schottenloher [23]). *Let* (X, ξ) *be a Riemann domain over a locally convex space* E, *and let* F *be a quasi-complete locally convex space. If* E *is a k-space, then the mapping*

$$f \in (H(X;F), \tau_0) \to S'_f \in (H(X), \tau_0)\varepsilon F$$

is a topological isomorphism. Its inverse is the mapping

$$T \in (H(X), \tau_0)\varepsilon F \to T \circ \varepsilon_X \in (H(X;F), \tau_0).$$

From Theorems 1.2 and 2.2 we obtain the following corollary.

Corollary 2.3 (Schottenloher [23]). *Let* (X, ξ) *be a Riemann domain over* E, *and assume that* E *is a k-space. Then the space* $(H(X), \tau_0)$ *has the approximation property if and only if*

$$(H(X;F), \tau_0) = \overline{H(X) \otimes F}$$

for every Banach space F, *or equivalently for every quasi-complete locally convex space* F.

Remark 2.4. A counterexample of Schottenloher [23, Example 1.4] shows that the conclusions of Lemma 2.1 and Theorem 2.2 may fail to be true without the k-space hypothesis.

3. Riemann domains over Fréchet spaces

Let (X, ξ) be a Riemann domain over a locally convex space E. If (Y, η) is another Riemann domain over E, then a *morphism* from (X, ξ) into (Y, η) is a continuous mapping $\tau : X \to Y$ such that $\eta \circ \tau = \xi$. If (Y, η) is a Riemann domain over F, and $T \in L(E;F)$, then a T-*morphism* is a continuous mapping $\tau : X \to Y$ such that $\eta \circ \tau = T \circ \xi$.

Let $cs(E)$ denote the family of all continuous seminorms on E. For $\alpha \in cs(E)$, $a, b \in E$ and $r > 0$, set

$$B_E^\alpha(a,r) = \{x \in E : \alpha(x-a) < r\},$$

$$\Delta_E(a,b,r) = \{a + \lambda b : \lambda \in \mathbf{C}, |\lambda| < r\}.$$

Consider the distance functions $d_X^\alpha : X \to [0, \infty]$ and $\delta_X : X \times E \to (0, \infty)$, which are defined as follows:

$$d_X^\alpha(x) = \sup\{r > 0: \text{ there is a section } \sigma : B_E^\alpha(\xi(x), r) \to X \text{ with } \sigma \circ \xi(x) = x\} \cup \{0\},$$

$$\delta_X(x,b) = \sup\{r > 0: \text{ there is a section } \sigma : \Delta_X(\xi(x), b, r) \to X \text{ with } \sigma \circ \xi(x) = x\}.$$

If $0 < r \leqslant d_X^\alpha(x)$, then $B_X^\alpha(x,r)$ denotes the connected component of $\xi^{-1}(B_E^\alpha(\xi(x),r))$ which contains x. Likewise, if $0 < r \leqslant \delta_X(x,b)$, then $\Delta_X(x,b,r)$ denotes the connected component of $\xi^{-1}(\Delta_E(\xi(x),b,r))$ which contains x.

If U is an open subset of E, then a function $f : U \to [-\infty, \infty)$ is said to be *plurisubharmonic* if it is upper semicontinuous and the function $\lambda \to f(a + \lambda b)$ is subharmonic on some open neighborhood of zero in $\mathbf{C}$ for each $a \in U$ and $b \in E$. $Ps(U)$ denotes the set of all plurisubharmonic functions on U.

If (X, ξ) is a Riemann domain over E, then a function $f : X \to [-\infty, \infty)$ is said to be *plurisubharmonic* if for each $x \in X$ there is a section $\sigma : \xi(x) + V \to X$, where $V \in \mathscr{V}(E)$ and $\sigma \circ \xi(x) = x$, such that $f \circ \sigma$ is plurisubharmonic on $\xi(x) + V$. $Ps(X)$ denotes the set of all plurisubharmonic functions on X. The domain X is said to be *pseudoconvex* if the function $-\log \delta_X$ is plurisubharmonic on $X \times E$.

If E has a Schauder basis (e_n), then E_n denotes the subspace generated by $e_1, \ldots, e_n$, and $T_n : E \to E_n$ denotes the canonical projection. If (X, ξ) is a Riemann domain over E, then we set $X_n = \xi^{-1}(E_n)$ and $\xi_n = \xi | E_n$. Then (X_n, ξ_n) is a Riemann domain over E_n.

The proof of Theorem 3.3 relies heavily on the next two lemmas, which summarize results of Dineen [6, Example 2.4] and Mujica [19, Lemmas 2.5–2.7 and 3.1]. We remark that the results in [19] sharpen results of Schottenloher [24], and are ultimately based on the approach to the Levi problem initiated by Gruman and Kiselman [11], and which became the model for all articles in that direction.

Lemma 3.1 (Mujica [19]). *Let E be a metrizable locally convex space with an equicontinuous Schauder basis and a continuous norm. Let (X, ξ) be a connected pseudoconvex Riemann domain over E. Then there are three increasing sequences (A_n), (B_n) and (C_n) of open subsets of X, and a sequence (τ_n) of mappings with the following properties*:

(a) *$C_n \subset B_n \subset A_n$ for every n, and $\bigcup_{n=1}^\infty C_n = X$.*
(b) *$\tau_n : A_n \to X_n$ is a T_n-morphism, $X_n \subset A_n$ and $\tau_n =$ identity on X_n for every n.*
(c) *$B_j \cap X_n \subset\subset A_j \cap X_n$ for every j and n.*

(d) $\tau_n(C_j) \subset B_j \cap X_n$ *whenever* $n \geqslant j$.

(e) *For each* $K \subset\subset X$ *and* $V \in \mathscr{V}(E)$ *such that* $K + V \subset X$, *there exists* $n_0 \in \mathbf{N}$ *such that* $K \subset C_n$ *and* $\tau_n(x) \in x + V$ *whenever* $x \in K$ *and* $n \geqslant n_0$.

(f) *For each* $f \in H(X_n)$ *there exists a sequence* $(g_k) \subset H(X)$ *which converges to* $f \circ \tau_n$ *uniformly on* C_n.

Lemma 3.2 (Dineen [6], Mujica [19]). *Let* E *be a locally convex space with an equicontinuous Schauder basis. Let* (X, ξ) *be a connected, pseudoconvex Riemann domain over* E, *and let* $x_0 \in X$. *Then there exists a directed, fundamental family of continuous seminorms* α *on* E *with the following properties*:

(a) $d_X^\alpha(x_0) > 0$ *and* $\alpha(x) = \sup_n \alpha(T_n x)$ *for every* $x \in E$.

(b) *There exists a complemented subspace* E_α *of* E *which has an equicontinuous Schauder basis and a continuous norm. More precisely* $E = E_\alpha \oplus \alpha^{-1}(0)$.

(c) *If we set* $X_\alpha = \xi^{-1}(E_\alpha)$ *and* $\xi_\alpha = \xi | X_\alpha$, *then* (X_α, ξ_α) *is a connected, pseudoconvex Riemann domain over* E_α.

(d) *If* $\pi_\alpha : E \to E_\alpha$ *denotes the canonical projection, then there exists a* π_α*-morphism* $\sigma_\alpha : X \to X_\alpha$ *such that* $\sigma_\alpha =$ *identity on* X_α.

(e) *If a function* $f \in H(X)$ *is bounded on an* α*-neighborhood of* x_0, *then* $f \circ \sigma_\alpha = f$ *on* X.

Now we can prove our main result for Riemann domains over Fréchet spaces.

Theorem 3.3. *Let* E *be a metrizable locally convex space with an equicontinuous Schauder basis, and let* (X, ξ) *be a connected, pseudoconvex Riemann domain over* E. *Then*:

(a) $H(X) \otimes F$ *is sequentially dense in* $(H(X; F), \tau_0)$ *for every Banach space* F.

(b) $H(X) \otimes F$ *is dense in* $(H(X; F), \tau_0)$ *for every quasi-complete locally convex space* F.

(c) $(H(X), \tau_0)$ *has the approximation property.*

Proof. (a) (i) First assume that E has a continuous norm, so that Lemma 3.1 applies. Given $f \in H(X; F)$, we see that $f \circ \tau_n \in H(A_n; F)$ for every n. We claim that $(f \circ \tau_n)$ converges to f uniformly on the compact subsets of X. Indeed given $K \subset\subset X$ and $\varepsilon > 0$, there exists $V \in \mathscr{V}(E)$ such that $K + V \subset X$ and $\|f(y) - f(x)\| < \varepsilon$ whenever $x \in K$ and $y \in x + V$. By Lemma 3.1(e) there exists $n_0 > 1/\varepsilon$ such that $K \subset C_n$ and $\tau_n(x) \in x + V$ whenever $x \in K$ and $n \geqslant n_0$. Hence

$$\|f \circ \tau_n(x) - f(x)\| < \varepsilon \tag{1}$$

whenever $x \in K$ and $n \geqslant n_0$. Consider the restriction $f | X_n \in H(X_n; F)$. Since E_n is finite dimensional, a result of Grothendieck [9, Chapitre II, p. 81] guarantees that $(H(X_n), \tau_0)$ is a nuclear Fréchet space and

$$(H(X_n; F), \tau_0) = \overline{H(X_n) \otimes F}.$$

Hence there exists $h_n \in H(X_n) \otimes F$ such that

$$||h_n(y) - f(y)|| < 1/n \tag{2}$$

for every $y \in B_n \cap X_n$. Write

$$h_n(y) = \sum_{i=1}^{p_n} h_{ni}(y) b_{ni},$$

with $h_{ni} \in H(X_n)$ and $b_{ni} \in F$ for $i = 1, \dots, p_n$. By Lemma 3.1(f) for each $i = 1, \dots, p_n$ there exists $g_{ni} \in H(X)$ such that

$$|g_{ni}(x) - h_{ni} \circ \tau_n(x)| < 1/np_n ||b_{ni}||$$

for every $x \in C_n$. If we define $g_n \in H(X) \otimes F$ by

$$g_n(x) = \sum_{i=1}^{p_n} g_{ni}(x) b_{ni},$$

then

$$||g_n(x) - h_n \circ \tau_n(x)|| < 1/n \tag{3}$$

for every $x \in C_n$. If $n \geqslant n_0$, then $\tau_n(K) \subset \tau_n(C_n) \subset B_n \cap X_n$. Hence for every $x \in K$ we have that

$$\begin{aligned} ||f(x) - g_n(x)|| \leqslant {} & ||f(x) - f \circ \tau_n(x)|| + ||f \circ \tau_n(x) - h_n \circ \tau_n(x)|| \\ & + ||h_n \circ \tau_n(x) - g_n(x)|| < 3\varepsilon. \end{aligned}$$

Thus (g_n) converges to f in $(H(X;F), \tau_0)$.

(ii) If E fails to have a continuous norm, then, given $f \in H(X;f)$, we fix $x_0 \in X$, and choose $\alpha \in cs(E)$ such that $\alpha(x) = \sup_n \alpha(T_n x)$ for every $x \in E$, $d_X^\alpha(x_0) > 0$, and f is bounded on some α-neighborhood of x_0. It follows from Lemma 3.2(e) that $f \circ \sigma_\alpha = f$ on X. Since E_α has an equicontinuous Schauder basis and a continuous norm, there exists a sequence $(h_n) \subset H(X_\alpha) \otimes F$ which converges to f in $(H(X_\alpha; F), \tau_0)$. Hence the sequence $(h_n \circ \tau_\alpha)$ lies in $H(X) \otimes F$, and converges to $f \circ \sigma_\alpha = f$ in $(H(X;F), \tau_0)$. This proves (a).

Conditions (b) and (c) follow from (a) by Corollary 2.3.

Remark 3.4. We used Corollary 2.3 to derive (b) from (a). It is not difficult to prove directly that if $H(X) \otimes F$ is dense in $(H(X;F), \tau_0)$ for every Banach space F, then $H(X) \otimes F$ is dense in $(H(X;F), \tau_0)$ for every complete locally convex space F. Indeed in this case F can be represented as a reduced projective limit of Banach spaces, and the proof is straightforward.

Corollary 3.5. *Let E be a separable Fréchet space with the bounded approximation property, and let (X, ξ) be a connected, pseudoconvex Riemann domain over E. Then:*

(a) *$H(X) \otimes F$ is sequentially dense in $(H(X;F), \tau_0)$ for every Banach space F.*

(b) *$H(X) \otimes F$ is dense in $(H(X;F), \tau_0)$ for every quasi-complete locally convex space F.*

(c) *$(H(X), \tau_0)$ has the approximation property.*

Proof. (a) By a result of Pelczynski (see the announcement in [22] or the detailed proof in [17]), there exist a Fréchet space M with a Schauder basis, and a Fréchet space N such that $M = E \times N$. Thus $(X \times N, (\xi, id_N))$ is a Riemann domain over $E \times N = M$. Let $\sigma : x \in X \to (x, 0) \in X \times N$, and let $\pi : (x, t) \in X \times N \to x \in X$. If $f \in H(X; F)$, then $f \circ \pi \in H(X \times N; F)$. By Theorem 3.3 there exists a sequence $(h_n) \subset H(X \times N) \otimes F$ which converges to $f \circ \pi$ in $(H(X \times N; F), \tau_0)$. Hence the sequence $(h_n \circ \sigma)$ lies in $H(X) \otimes F$, and converges to $f \circ \pi \circ \sigma = f$ in $(H(X; F), \tau_0)$. This proves (a).

Conditions (b) and (c) follow from (a) by Corollary 2.3.

Before removing the hypothesis of pseudoconvexity in Corollary 3.5 we need some preparation. Let (X, ξ) be a connected Riemann domain over a quasi-complete locally convex space E, and consider the spectrum $S(H(X), \tau_0)$, that is the set of all nonzero continuous algebra homomorphisms $T : (H(X), \tau_0) \to \mathbf{C}$. By the Mackey–Arens theorem there is a mapping

$$\pi_X : S(H(X), \tau_0) \to E$$

such that $\phi \circ \pi_X(T) = T(\phi \circ \xi)$ for every $T \in S(H(X), \tau_0)$ and $\phi \in E'$. Let

$$\varepsilon_X : x \in X \to \varepsilon_x \in S(H(X), \tau_0)$$

denote the evaluation mapping, that is $\varepsilon_x(f) = f(x)$ for every $f \in H(X)$. For each $f \in H(X)$ let $\hat{f} : S(H(X), \tau_0) \to \mathbf{C}$ be defined by $\hat{f}(T) = T(f)$ for every $T \in S(H(X), \tau_0)$. Then we have the following theorem, which is essentially due to Alexander [1] and Schottenloher [25].

Theorem 3.6 (Alexander [1], Schottenloher [25]). *Let (X, ξ) be a connected Riemann domain over a quasi-complete locally convex space E. Then there is a Hausdorff topology on $S(H(X), \tau_0)$ with the following properties:*

(a) *$(S(H(X), \tau_0), \pi_X)$ is a pseudoconvex Riemann domain over E.*
(b) *The mapping $\varepsilon_X : X \to S(H(X), \tau_0)$ is a morphism.*
(c) *If $\hat{X}$ denotes the connected component of $S(H(X), \tau_0)$ which contains $\varepsilon_X(X)$, then $\hat{f} \in H(\hat{X})$ for every $f \in H(X)$, and the extension mapping*
$$f \in (H(X), \tau_0) \to \hat{f} \in (H(\hat{X}), \tau_0)$$
is a topological isomorphism.
(d) *If F is a quasi-complete locally convex space, then each $f \in H(X; F)$ admits an extension $\hat{f} \in H(\hat{X}; F)$, and the extension mapping*
$$f \in (H(X; F), \tau_0) \to \hat{f} \in (H(\hat{X}; F), \tau_0)$$
is a topological isomorphism.

Proof. This theorem is essentially due to Alexander [1] in the case of Banach spaces, and to Schottenloher [25] in the case of locally convex spaces. Since (d) is not mentioned in [25], and is only stated without proof in [23, Remark 4.4], we include a proof here for the sake of completeness.

If E is a k-space, then (d) follows easily from (c) with the aid of Theorem 2.2. Indeed by (c) the extension mapping $(H(X), \tau_0) \to (H(\hat{X}), \tau_0)$ is a topological isomorphism. Then by using Theorem 2.2 we obtain the topological isomorphisms

$$(H(X;F), \tau_0) = (H(X), \tau_0)\varepsilon F = (H(\hat{X}), \tau_0)\varepsilon F = (H(\hat{X};F), \tau_0).$$

An examination of the mappings in Theorem 2.2 shows that the topological isomorphism $(H(X;F), \tau_0) = (H(\hat{X};F), \tau_0)$ thus obtained is precisely the extension mapping $H(X;F) \to H(\hat{X};F)$.

If E is an arbitrary locally convex space, we proceed as follows. The proof of Alexander [1, Section 4, Theorem 2] shows that it follows from (c) that every $f \in H(X;F)$ admits an extension $\hat{f} : \hat{X} \to F$, which is weakly holomorphic, and therefore Gateaux holomorphic. (This follows also from a result of Bogdanowicz [5, Corollary 3], which became the starting point of most articles on vector-valued holomorphic continuation.) To prove that $\hat{f}$ is continuous, we may assume, without loss of generality, that F is a Banach space. If $T \in \hat{X}$, then there is a compact set $K \subset X$ such that

$$|\hat{\phi}(T)| = |T(\phi)| \leqslant ||\phi||_K \tag{4}$$

for every $\phi \in H(X)$. Since f is continuous, there is $V \in \mathscr{V}(E)$ such that $K + 2V \subset X$ and $||f||_{K+2V} < \infty$. Let $\sum_{m=0}^{\infty} P^m\phi(x)(t) = \phi(x+t)$ be the Taylor series expansion of $\phi \in H(X)$ at $x \in X$, and set $P_t^m\phi(x) = P^m\phi(x)(t)$ for every $m \in \mathbf{N}$, $x \in X$ and $t \in E$. Then $P_t^m\phi \in H(X)$ and $(P_t^m\phi)^\wedge = P_t^m\hat{\phi}$ for every $m \in \mathbf{N}$ and $t \in E$. It follows from (4) and the Cauchy inequalities that

$$|P_t^m\hat{\phi}(T)| = |(P_t^m\phi)^\wedge(T)| \leqslant ||P_t^m\phi||_K \leqslant 2^{-m}||\phi||_{K+2V} \tag{5}$$

for every $m \in \mathbf{N}$, $T \in \hat{X}$ and $t \in V$. By applying (5) to $\phi = \psi \circ f$, with $\psi \in F'$, $||\psi|| \leqslant 1$, it follows that

$$|\psi \circ P_t^m\hat{f}(T)| = |P_t^m(\psi \circ \hat{f})(T)| = |P_t^m(\psi \circ f)^\wedge(T)| \leqslant 2^{-m}||\psi \circ f||_{K+2V},$$

and therefore

$$||P_t^m\hat{f}(T)|| \leqslant 2^{-m}||f||_{K+2V} \tag{6}$$

for every $m \in \mathbf{N}$, $T \in \hat{X}$ and $t \in V$. It follows from (6) that each $P^m\hat{f}(T)$ is continuous, and the series $\sum_{m=0}^{\infty} P^m\hat{f}(T)(t)$ converges uniformly for $t \in V$. Hence it follows that $\hat{f}$ is continuous.

Since the extension mapping $(H(X), \tau_0) \to (H(\hat{X}), \tau_0)$ is continuous, for each compact set $L \subset \hat{X}$, there are a compact set $K \subset X$ and $c > 0$ such that $||\hat{\phi}||_L \leqslant c||\phi||_K$ for every $\phi \in H(X)$. By applying this inequality to $\psi \circ f$, with $\psi \in F'$, $||\psi|| \leqslant 1$, it follows that $||\hat{f}||_L \leqslant c||f||_K$ for every $f \in H(X;F)$. Hence the extension mapping $(H(X;F), \tau_0) \to (H(\hat{X};F), \tau_0)$ is also continuous, and the proof of (d) is complete. □

Corollary 3.5 and Theorem 3.6 immediately yield the following corollary.

Corollary 3.7. *Let E be a separable Fréchet space with the bounded approximation property, and let (X, ξ) be a connected Riemann domain over E. Then*:

(a) *$H(X) \otimes F$ is sequentially dense in $(H(X;F), \tau_0)$ for every Banach space F.*

(b) *$H(X) \otimes F$ is dense in $(H(X;F), \tau_0)$ for every quasi-complete locally convex space F.*

(c) *$(H(X), \tau_0)$ has the approximation property.*

As an application of our results we obtain the following theorem.

Theorem 3.8. *Let E be a separable Fréchet space with the bounded approximation property, let (X, ξ) be a connected Riemann domain over E, and let A be a complete locally m-convex algebra with a unit element. Then*:

(a) *The spectrum $S(H(X;A), \tau_0)$ can be canonically identified with $S(H(X), \tau_0) \times S(A)$.*

(b) *If X is pseudoconvex, then the spectrum $S(H(X;A), \tau_0)$ can be canonically identified with $X \times S(A)$.*

Proof. (a) Given $(T_1, T_2) \in S(H(X), \tau_0) \times S(A)$, it is clear that the formula

$$Tf = T_1(T_2 \circ f) \quad \text{for every } f \in H(X;A) \tag{7}$$

defines a $T \in S(H(X;A), \tau_0)$. Conversely, we will show that every $T \in S(H(X;A), \tau_0)$ is of the form (7). Indeed given $T \in S(H(X;A), \tau_0)$, it follows that each of the functions

$$\phi \in (H(X), \tau_0) \to T(\phi \otimes 1) \in \mathbf{C}$$

and

$$y \in A \to T(1 \otimes y) \in \mathbf{C}$$

is a continuous algebra homomorphism. Hence there are $T_1 \in S(H(X), \tau_0)$ and $T_2 \in S(A)$ such that

$$T(\phi \otimes 1) = T_1(\phi) \quad \text{for every } \phi \in H(X)$$

and

$$T(1 \otimes y) = T_2(y) \quad \text{for every } y \in A.$$

Since $\phi \otimes y = (\phi \otimes 1)(1 \otimes y)$, it follows that

$$T(\phi \otimes y) = T_1(\phi) T_2(y) = T_1(T_2 \circ (\phi \otimes y))$$

for every $\phi \in H(X)$ and $y \in A$. Hence (4) holds for every $f \in H(X) \otimes A$. Since $H(X) \otimes A$ is dense in $(H(X;A), \tau_0)$, by Corollary 3.7, it follows that (7) holds for every $f \in H(X;A)$.

(b) It suffices to apply (a), in tandem with a theorem of Schottenloher [25], which asserts that the spectrum $S(H(X), \tau_0)$ can be canonically identified with X. □

4. Riemann domains over (DFC)-spaces

In this section we prove that if (X, ξ) is a connected, pseudoconvex Riemann domain over a (DFC)-space with the approximation property, then $(H(X), \tau_0)$ has the approximation property. We will derive this result from Theorem 3.3 with the aid of results of Lourenço [15,16]. Before stating Lourenço's results in the most convenient way we need some preparation.

If E is a locally convex space, and $\alpha \in cs(E)$, then (E, α) denotes the vector space E, seminormed by α, and E_α denotes the normed space $(E, \alpha)/\alpha^{-1}(0)$. Let $\pi_\alpha \in L(E, E_\alpha)$ denote the canonical mapping.

We recall that E is said to be a (DFC)-space if $E = D'_c$, for a suitable Fréchet space D. (DFC)-spaces were introduced and studied by Hölstein [12,13] in connection with the theory of topological tensor products. For the theory of holomorphic functions on (DFC)-spaces we refer to Mujica [18], Valdivia [29], Schottenloher [26], Nachbin [21], Lourenço [15,16] and Galindo et al. [9]. It follows from the Banach–Dieudonné theorem that every (DFC)-space is a k-space (see [18] or [29]). Then we have the following lemma, which is essentially due to Schottenloher [26]. With the exception of (a), the results below are never stated explicitly, but they are implicit in [26].

Lemma 4.1 (Schottenloher [26]). *Let E be a (DFC)-space, and let (X, ξ) be a connected, pseudoconvex Riemann domain over E. Then:*

(a) *X is hemicompact.*

(b) *There exists $\alpha \in cs(E)$ such that $d_X^\alpha(x) > 0$ for every $x \in X$, there exists a connected, pseudoconvex Riemann domain (X_α, ξ_α) over E_α, and there exists a π_α-morphism $\pi_\alpha^* : X \to X_\alpha$.*

(c) *For each $f \in H(X; F)$, where F is a Banach space, we may choose the seminorm α and the domain X_α in* (b) *in such a way that $f = f_\alpha \circ \pi_\alpha$, with $f_\alpha \in H(X_\alpha; F)$.*

Proof. Condition (a) is [26, Lemma 7]. By using results of Mujica [18] and Schottenloher [24], and a standard factorization method, we prove (b) and (c) at the same time. Let (K_n) be an increasing, fundamental sequence of compact subsets of X, and let $f \in H(X; F)$, where F is a Banach space. For each n there is $\alpha_n \in cs(E)$ such that $K_n + B_E^{\alpha_n}(0, 1) \subset X$ and f is bounded on $K_n + B_E^{\alpha_n}(0, 1)$. By Mujica [18, Corollary 7.9] there is $\alpha \in cs(E)$ such that $\alpha \geqslant r_n \alpha_n$, with $r_n > 0$, for every n. In particular $d_X^\alpha(x) > 0$ and f is bounded on $B_X^\alpha(x; r_x)$, where $r_x > 0$, for every $x \in X$. By Schottenloher [24, Lemma 1.7] $\delta_X(x, b) = \infty$ for every $x \in X$ and $b \in \alpha^{-1}(0)$. By Schottenloher [24, Proposition 1.8] there exists a connected, pseudoconvex Riemann domain $X/\alpha^{-1}(0)$ over the quotient space $E/\alpha^{-1}(0)$, and there exists a π_α-morphism $\pi_\alpha^* : X \to X/\alpha^{-1}(0)$. The space $X/\alpha^{-1}(0)$ is the quotient of X under the equivalence relation $\sim$ defined by $y \sim x$ if $y \in \Delta_X(x, b, \infty)$ for some $b \in \alpha^{-1}(0)$. Let X_α denote the set $X/\alpha^{-1}(0)$, with the unique topology which makes it a Riemann domain over the normed space E_α. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} X & \overset{\pi_\alpha^*}{\rightarrow} & X/\alpha^{-1}(0) & \overset{id}{\rightarrow} & X_\alpha \\ \xi\downarrow & & \downarrow\xi_\alpha & & \downarrow\xi_\alpha \\ E & \overset{\pi_\alpha}{\rightarrow} & E/\alpha^{-1}(0) & \overset{id}{\rightarrow} & E_\alpha \end{array}$$

Since f is bounded on $B_X^\alpha(x, r_x)$, with $r_x > 0$, for every $x \in X$, it follows that f is bounded on $\Delta_X(x, b, \infty)$ for every $x \in X$ and $b \in \alpha^{-1}(0)$. It follows from Liouville's theorem that f is constant on $\Delta_X(x, b, \infty)$ for every $x \in X$ and $b \in \alpha^{-1}(0)$. Thus $f = f_\alpha \circ \pi_\alpha^*$, with $f_\alpha \in H(X_\alpha; F)$, as asserted. □

If (X, ξ) is a connected, pseudoconvex Riemann domain over a (DFC)-space E, then $\mathscr{A}(X)$ denotes the family of all $\alpha \in cs(E)$ which verify (b) in Lemma 4.1. Then we have the following factorization theorem, which is essentially due to Lourenço [15,16].

Theorem 4.2 (Lourenço [15,16]). *Let E be a (DFC)-space with the approximation property, and let (X, ξ) be a connected, pseudoconvex Riemann domain over E. Then:*

(a) *E is the projective limit of a family $(G_i)_{i \in I}$ of normed spaces, each of which has an equicontinuous Schauder basis. Furthermore, for each $\alpha \in cs(E)$, there are $i \in I$ and $C_i \in L(G_i; E_\alpha)$ such that $C_i \circ \sigma_i = \pi_\alpha$, where $\sigma_i \in L(E, G_i)$ denotes the canonical mapping.*

(b) *For each $\alpha \in \mathscr{A}(X)$, there exists a connected, pseudoconvex Riemann domain (X_i, ξ_i) over some G_i, there exists a σ_i-morphism $\sigma_i^* : X \to X_i$, and there exists a C_i-morphism $C_i^* : X_i \to X_\alpha$ such that $C_i^* \circ \sigma_i^* = \pi_\alpha^*$.*

(c) *For each $f \in H(X; F)$, where F is a Banach space, we may choose the domain (X_i, ξ_i) in* (b) *in such a way that $f = f_i \circ \sigma_i^*$, with $f_i \in H(X_i; F)$.*

Proof. Condition (a) is proved in [15, Theorem 2.1]. Condition (b) was proved in [16, Theorem 1.1] when E has a continuous norm, and in that case the Riemann domain (X_i, σ_i) has some additional properties. An examination of the proof in [16] shows that the hypothesis of the continuous norm was used only to establish those additional properties. Condition (c) was not stated explicitly in [16], but follows at once from (b) with the aid of Lemma 4.1(c). □

With the notation of Theorem 4.2 we have the following commutative diagram:

$$\begin{array}{ccccc} X & & \overset{\pi_\alpha^*}{\rightarrow} & & X_\alpha \\ \sigma_i^* & \searrow & & \nearrow & C_i^* \\ \xi\downarrow & & X_i & & \downarrow\xi_\alpha \\ E & & \overset{\pi_\alpha}{\rightarrow} & & E_\alpha \\ \sigma_i & \searrow & \downarrow\xi_i & \nearrow & C_i \\ & & G_i & & \end{array}$$

Now we can prove our main result for Riemann domains over (DFC)-spaces.

Theorem 4.3. *Let E be a* (*DFC*)*-space with the approximation property, and let* (X, ξ) *be a connected, pseudoconvex Riemann domain over E. Then*:

(a) $H(X) \otimes F$ *is* (*sequentially*) *dense in* $(H(X;F), \tau_0)$ *for every Banach space F.*

(b) $H(X) \otimes F$ *is dense in* $(H(X;F), \tau_0)$ *for every quasi-complete locally convex space F.*

(c) $(H(X), \tau_0)$ *has the approximation property.*

Proof. (a) Let $f \in H(X;F)$. With the notation of Theorem 4.2 we have that $f = f_i \circ \sigma_i^*$, where $f_i \in H(X_i;F)$. By Theorem 3.3 there is a sequence $(h_n) \subset H(X_i) \otimes F$ which converges to f_i in $(H(X_i;F), \tau_0)$. Hence the sequence $(h_n \circ \sigma_i^*)$ lies in $H(X) \otimes F$ and converges to $f_i \circ \sigma_i^* = f$ in $(H(X;F), \tau_0)$. This shows (a).

Conditions (b) and (c) follow from (a) by Corollary 2.3.

Corollary 4.4. *Let* (X, ξ) *be a connected Riemann domain over a* (*DFC*)*-space E. Then the following conditions are equivalent*:

(1) *E has the approximation property.*

(2) $H(X) \otimes F$ *is* (*sequentially*) *dense in* $(H(X;F), \tau_0)$ *for every Banach space F.*

(3) $H(X) \otimes F$ *is dense in* $(H(X;F), \tau_0)$ *for every quasi-complete locally convex space F.*

(4) $(H(X), \tau_0)$ *has the approximation property.*

Proof. Implication (1) ⇒ (2) follows from Theorems 3.6 and 4.3. Implications (2) ⇒ (3) and (3) ⇒ (4) follow from Corollary 2.3. Since E'_c is a complemented subspace of $(H(X), \tau_0)$, it follows from (4) that E'_c has the approximation property, and therefore E has the approximation property, by Corollary 1.3. Thus (4) ⇒ (1).

Schottenloher [23] remarked that $(C(X), \tau_0)$ has the approximation property for every k-space X, and raised the question as to whether $(H(U), \tau_0)$ has the approximation property for every open subset U of a locally convex space E, whenever E is a k-space with the approximation property. Corollaries 3.7 and 4.4 provide partial answers to that question.

As an application of our results we obtain the following theorem.

Theorem 4.5. *Let E be a* (*DFC*)*-space with the approximation property, let* (X, ξ) *be a connected Riemann domain over E, and let A be a complete locally m-convex algebra with a unit element. Then*:

(a) *The spectrum* $S(H(X;A), \tau_0)$ *can be canonically identified with* $S(H(X), \tau_0) \times S(A)$.
(b) *If X is pseudoconvex, then the spectrum* $S(H(X;A), \tau_0)$ *can be canonically identified with* $X \times S(A)$.

Proof. We can derive (a) from Corollary 4.4 in the same way we derived Theorem 3.8(a) from Corollary 3.7. And (b) follows from (a) with the aid of another result of

Schottenloher [26, Proposition 4], which asserts that the spectrum $S(H(X), \tau_0)$ can be canonically identified with X. $\square$

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